Indian J. Phys. **44**, 399–404, (1970)

Dielectric function of degenerate plasma at relativistic temperatures

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(*Received* 12 *May* 1970)

Expressions for the transverse and longitudinal dielectric functions of a degenerate plasma are obtained at relativistic temperatures The nature of wave propagation is discussed and results compared to those obtained for a degenerate plasma at non-relativistic temperatures (Misra *et al* 1962, 1970).

INTRODUCTION

Plasmas occuring in stars are at very high temperatures. The temperature of some of the stars is about 10^8 °K or even more For such stars the charged particles constituting the plasma move with very high velocities. It was pointed out by Fowler (1926) that the density of particles in some of these stars (white dwarfs) is so high that even at the relativistic temperatures the plasma is considered as degenerate. Chandrasekhar (1957) studied the properties of such stars using relativistic Fermi-Dirac distribution law and evaluated the integrals using Sommerfeld's (1928) approximate method. Buti (1963) studied the properties of thermonuclear (non-degenerate) plasma using relativistic Boltzmann-Vlassov equation and Maxwell distribution law. Here we use relativistic Boltzmann-Vlasov equation with Fermi-Dirac equilibrium distribution law to obtain explicit expressions for dielectric functions of a degenerate relativistic plasma, evaluating the integrals by Sommerfeld's approximate method.

DIELECTRIC FUNCTION FROM RELATIVISTIC BOLTZMANN-VLASSOV EQUATION

In this section we shall derive formulae for the dielectric function of a relativistic plasma. The relativistic Boltzmann-Vlassov equation is

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t}+\frac{c\vec{p}}{(p^2+m^2c^2)^{\frac{1}{2}}}\cdot\vec{\nabla}_r f(\vec{r},\vec{p},t)-e\vec{E}(\vec{r},t)\cdot\vec{\nabla}_p f=0 \qquad \ldots \quad (1)$$

writing $f(\vec{r},\vec{p},t)=n_0f_0(\vec{p})+f_1(\vec{r},\vec{p},t)$ we obtain the following linearized relativistic B. V. equation :

$$\frac{\partial f_1(\vec{r},\vec{p},t)}{\partial t}+\frac{c\vec{p}}{(p^2+m^2c^2)^{\frac{1}{2}}}\cdot\vec{\nabla}_r f_1(\vec{r},\vec{p},t)$$
$$=n_0e\vec{E}(\vec{r},t)\cdot\vec{\nabla}_p f_0(\vec{p}) \qquad \ldots \quad (2)$$

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A solution is easily obtained by taking the momentum Fourier transform in polar co-ordinates (p, θ, ϕ) for $\vec{p}$

$$f_1(\vec{k}, \vec{p}, \omega) = \frac{ien_0 \frac{df_0}{dp}(p^2+m^2c^2)^{\frac{1}{2}}}{\omega(p^2+m^2c^2)^{\frac{1}{2}}-cpk\cos\theta}$$

$$\times[E_1 \sin\theta\cos\phi+E_2\sin\theta\sin\phi+E_3\cos\theta] \quad \ldots \quad (3)$$

The wave number and frequency dependent current density can be expressed as (Gartenhaus 1964)

$$J_\alpha(\vec{k}, \omega) = -ec\int \vec{dp}\, \frac{p}{(p^2+m^2c^2)^{\frac{1}{2}}} f_1(\vec{k}, \vec{p}, \omega)$$

$$= K_{\alpha\beta}(\vec{k}, \omega)E_\beta(\vec{k}, \omega) \quad \ldots \quad (4)$$

Using equation (3) in equation (4) we get the following expressions for the transverse and longitudinal response functions

$$K_{tr}(\vec{k}, \omega) = -\frac{i\omega_0^2 cm\omega}{2}\int_0^\infty\int_0^1 \frac{p^3 dp dx (p^2+m^2c^2)^{\frac{1}{2}}(1-x^2)\frac{df_0}{dp}}{\omega^2(p^2+m^2c^2)-c^2p^2k^2x^2} \quad \ldots \quad (5)$$

and

$$K_l(\vec{k}, \omega) = -i\omega_0^2 cm\omega\int_0^\infty\int_0^1 \frac{p^3 dp dx (p^2+m^2c^2)^{\frac{1}{2}}x^2\frac{df_0}{dp}}{\omega^2(p^2+m^2c^2)-c^2p^2k^2x^2} \quad \ldots \quad (6)$$

where $\omega_0^2 = \frac{4\pi n_0 e^2}{m}$ and $x = \cos\theta$

Computation of Dielectric Function for Relativistic Fermi-Dirac Distribution

For relativistic Fermi distribution we shall use

$$n_0 f_0 = \frac{2}{h^3}\frac{1}{\frac{1}{\Lambda}\exp\left[\frac{c}{KT}(p^2+m^2c^2)^{\frac{1}{2}}\right]+1} \quad \ldots \quad (7)$$

where $\frac{1}{\Lambda} = \exp\left[-\left(\nu+\frac{mc^2}{KT}\right)\right]$

Expression (5) directly gives the imaginary part of the response function $K_{tr}(k, \omega)$;

$$\mathrm{Im}K_{tr}(\vec{k}, \omega) = \frac{\omega_0^2 zm^3c\omega}{2k^2n_0h^3\Lambda}\int_0^\infty \frac{d\theta\,\mathrm{sh}\,\theta\exp(Z\,\mathrm{ch}\theta)}{\left[\frac{1}{\Lambda}\exp(Z\,\mathrm{ch}\,\theta)+1\right]^2}\times$$

$$\left[\mathrm{sh}\,2\theta+\frac{c^2k^2\,\mathrm{sh}^2\theta-\omega^2\mathrm{ch}^2\theta}{\omega ck}\ln\left|\frac{\omega\,\mathrm{ch}\,\theta+ck\,\mathrm{sh}\,\theta}{\omega\mathrm{ch}\,\theta-ck\,\mathrm{sh}\,\theta}\right|\right] \quad \ldots \quad (8)$$

After integrating by parts we get

$$\mathrm{Im}K_{tr}(\vec{k}, \omega) = \frac{\omega_0{}^2 m^3 c\omega}{k^2 n_0 h^3 Z} \int_0^{\frac{1}{\Lambda}} \frac{du}{\exp(u)+1} \frac{d\phi(u)}{du}$$

$$= \frac{\omega_0{}^2 m^3 c\omega}{k^2 n_0 h^3 Z} I \tag{9}$$

where
$$\frac{d\phi}{du} = \frac{1}{\mathrm{sh}\,\theta}\left[\mathrm{ch}2\theta + \frac{c^2k^2-\omega^2}{\omega ck}\,\mathrm{sh}2\theta \times \ln\left|\frac{\omega\,\mathrm{ch}\,\theta + ck\,\mathrm{sh}\,\theta}{\omega\,\mathrm{ch}\,\theta - ck\,\mathrm{sh}\,\theta}\right| - 1\right] \quad \ldots \tag{10}$$

with

$$u = Z\mathrm{ch}\theta.$$

The integral contained in equation (9) is difficult to evaluate However, in problems of similar nature a method due to Sommerfeld is extensively used. One expands the integrand round a *stable* value $u_0 = Z\,\mathrm{ch}\theta_0$ as

$$I = \phi(u_0) + 2c_2\phi''(u_0) \tag{11}$$

where
$$c_2 = \frac{\pi^2}{12}.$$

The higher terms of the series cannot be neglected if the ratio of the successive terms is greater than one. We shall study the typical cases at high temperatures when the series converges much faster. From equation (10) we get

$$\phi(u_0) = Z\left[\frac{\mathrm{sh}2\theta_0}{2} + \frac{c^2k^2\mathrm{sh}^2\theta_0 - \omega^2\mathrm{ch}^2\theta_0}{2ck\omega} \times \ln\left|\frac{\omega\mathrm{ch}\,\theta_0 + ck\,\mathrm{sh}\,\theta_0}{\omega\mathrm{ch}\theta_0 - ck\,\mathrm{sh}\,\theta_0}\right|\right] \quad \ldots \tag{12}$$

and

$$\left(\frac{d^2\phi}{du^2}\right)_{u=u_0} = \frac{c^2k^2-\omega^2}{Zck\,\omega}\ln\left|\frac{\omega\mathrm{ch}\,\theta_0 + ck\,\mathrm{sh}\,\theta_0}{\omega\mathrm{ch}\theta_0 - ck\,\mathrm{sh}\,\theta_0}\right| + \frac{2\mathrm{ch}\theta_0}{Z\,\mathrm{sh}\,\theta_0}\left[1 + \frac{c^2k^2-\omega^2}{\omega^2\mathrm{ch}^2\theta_0 - c^2k^2\,\mathrm{sh}^2\,\theta_0}\right] \quad \ldots \tag{13}$$

The response function is, from equations (11). (12) and (13)

$$\operatorname{Im} K_{tr}(\vec{k}, \omega) = \frac{\omega_0^2 m^3 c\omega}{n_0 h^3 k^2} \left[\frac{\operatorname{sh} 2\theta_0}{2} + \frac{c^2k^2 \operatorname{sh}^2\theta_0 - \omega^2 \operatorname{ch}^2 \theta_0}{2\omega ck} \times \ln \left| \frac{\omega \operatorname{ch} \theta_0 + ck \operatorname{sh} \theta_0}{\omega \operatorname{ch} \theta_0 - ck \operatorname{sh} \theta_0} \right| \right.$$

$$+ \frac{\omega_0^2 m^3 c\pi^2 \omega}{6 n_0 h^3 k^2 Z^2} \left[\frac{c^2k^2 - \omega^2}{\omega ck} \ln \left| \frac{\omega \operatorname{ch} \theta_0 + ck \operatorname{sh} \theta_0}{\omega \operatorname{ch} \theta_0 - ck \operatorname{sh} \theta_0} \right| \right.$$

$$\left. + \frac{2 \operatorname{ch} \theta_0}{\operatorname{sh} \theta_0} \left(1 + \frac{c^2k^2 - \omega^2}{\omega^2 \operatorname{ch}^2 \theta_0 - c^2k^2 \operatorname{sh}^2 \theta_0} \right) \right] \quad \ldots \quad (14)$$

As already noted the formula (11) can be used only when the second term is small compared to the first. This corresponds to the condition

$$\pi^2 \left[\frac{c^2k^2 - \omega^2}{ck\omega} \ln \left| \frac{\omega \operatorname{ch} \theta_0 + ck \operatorname{sh} \theta_0}{\omega \operatorname{ch} \theta_0 - ck \operatorname{sh} \theta_0} \right| + \frac{2 \operatorname{ch} \theta_0}{\operatorname{sh} \theta_0} \left(1 + \frac{c^2k^2 - \omega^2}{\omega^2 \operatorname{ch}^2 \theta_0 - c^2k^2 \operatorname{sh}^2 \theta_0} \right) \right]$$

$$\leqslant 3Z^2 \left[\operatorname{sh} 2\theta_0 + \frac{c^2k^2 \operatorname{sh}^2 \theta_0 - \omega^2 \operatorname{ch}^2 \theta_0}{\omega ck} \ln \left| \frac{\omega \operatorname{ch} \theta_0 + ck \operatorname{sh} \theta_0}{\omega \operatorname{ch} \theta_0 - ck \operatorname{sh} \theta_0} \right| \right] \quad (15)$$

where θ_0 is related to the Fermi momentum by the relation

$$\operatorname{sh} \theta_0 = p_0/mc.$$

The condition given in equation (15) is satisfied if

$$p_0 \gg mc$$

provided the temperature is such that

$$\frac{\pi^2 K^2 T^2}{3c^2 p_0^2} < 1$$

For a relativistic plasma for which $T > 10^8$ °K the above condition is fulfilled if the plasma is very dense, *i e.*

$$n_0 \gg 10^{27}.$$

In other words the plasma becomes highly degenerate. For such a degenerate plasma with $p_0 \gg mc$, equation (14) simplifies to

$$\operatorname{Im} K_{tr}(\vec{k}, \omega) = \frac{3\omega_0^2 \omega^2}{16\pi c^2 k^3 v_0} \left(1 + \frac{\pi^2 K^2 T^2}{3c^2 m^2 v_0^2} \right) \left[\frac{2ck}{\omega} + \frac{c^2k^2 - \omega^2}{\omega^2} \ln \left| \frac{\omega + ck}{\omega - ck} \right| \right]$$

from which we obtain

$$\epsilon_{tr}(\vec{k}, \omega) = 1 - \frac{3\omega_0^2 \omega}{4c^2 k^3 v_0} \left(1 + \frac{\pi^2 K^2 T^2}{3c^2 m^2 v_0^2} \right) \times$$

$$\left[\frac{2ck}{\omega} + \frac{c^2k^2 - \omega^2}{\omega^2} \ln \left| \frac{\omega + ck}{\omega - ck} \right| \right] \quad \ldots \quad (16)$$

An expression for the longitudinal dielectric constant can be obtained by performing analogous computation for $K_l(\vec{k}, \omega)$. We shall not go into the details but give the final result as

$$\epsilon_l(\vec{k}, \omega) = 1+\frac{3\omega_0^2\omega}{2c^2k^3v_0}\left[1+\frac{\pi^2K^2T^2}{3m^2c^2v_0^2}\right]\times\left[\frac{2ck}{\omega}-\ln\left|\frac{\omega+ck}{\omega-ck}\right|\right] \quad \ldots \quad (17)$$

When the effect of temperature is neglected, the above expressions for the dielectric constants reduce to that obtained by Silin (1960)

Nature of Wave Propagation

The analysis of wave propagation in a relativistic plasma is possible with the aid of an expression for the dielectric constant derived in the previous section. Certain interesting conclusions can be drawn without involving ourselves in complex algebraic manipulations, for certain limiting cases

Case I

$$\frac{ck}{\omega} < 1 \quad i.e., \quad n \simeq \epsilon^{\frac{1}{2}} < 1$$

Under this condition ln term in equations (16) and (17) can be expanded in a power series of n and after simple manipulation we get

$$\epsilon_{tr}(\vec{k}, \omega) = \frac{1-\dfrac{\omega_0^2c}{v_0\omega^2}\left[1+\dfrac{\pi^2K^2T^2}{3c^2m^2v_0^2}\right]}{1+\dfrac{\omega_0^2c}{5\omega^2v_0}\left[1+\dfrac{\pi^2K^2T^2}{3c^2m^2v_0^2}\right]}$$

and

$$\epsilon_l(\vec{k}, \omega) = \frac{1-\dfrac{\omega_0^2c}{\omega^2v_0}\left[1+\dfrac{\pi^2K^2T^2}{3c^2m^2v_0^2}\right]}{1+\dfrac{\omega_0^2c}{5\omega^2v_0}\left[1+\dfrac{\pi^2K^2T^2}{3c^2m^2v_0^2}\right]}$$

It will be seen from the above expressions that both transverse and longitudinal waves can propagate in the plasma under consideration, if their frequency is greater than a limiting frequency, *i e* if

$$\omega > \omega_0\left[\frac{c}{v_0}\left(1+\frac{\pi^2K^2T^2}{3c^2m^2v_0^2}\right)\right]^{1/2} \quad \ldots \quad (18)$$

This is to be contrasted with that of wave propagation in a non-relativistic degenerate plasma for which wave propagation (Pradhan *et al* 1960, Misra *et al* 1962 and Misra *et al* 1970) is possible if

$$\omega > \omega_0\left(1+\frac{3\pi^2K^2T^2}{4m^2v_0^4}\right)^{1/2} \quad \ldots \quad (19)$$

It is interesting to note that if we put $c = v_0$ in the equation (18) we get approximately the corresponding expression obtained for non-relativistic case,

Case II

$$\frac{ck}{\omega} > 1$$

$$i.e, \qquad \epsilon_l^{\frac{1}{2}} \simeq n_l > 1$$

Under this conditions equations (16) and (17) take the simple forms,

$$\epsilon_{tr}(\epsilon_{tr}-1) = -\frac{3\omega_0{}^2c}{\omega^2 v_0}\left(1+\frac{\pi^2K^2T^2}{3m^2c^2v_0{}^2}\right)$$

and

$$\epsilon_l(\epsilon_l-1) = \frac{3\omega_0{}^2c}{\omega^2 v_0}\left(1+\frac{\pi^2K^2T^2}{3m^2c^2v^2{}_0}\right)$$

It will be seen from these equations that longitudinal waves can propagate through the relativistic plasma, whereas, transverse waves cannot do so for $n < 1$. Similar results can be obtained for a degenerate non-relativistic plasma. (Misra *et al* 1962).

Acknowledgement

The authors are grateful to Prof T. Pradhan of Saha Institute of Nuclear Physics, Calcutta, for suggesting the problem and for guidance. They would also like to thank Sri B. C Roy, Department of Physics, Ravenshaw College for help and comments while the work was in progress.

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